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Multigrid for the Galerkin least squares method in linear elasticity

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Abstract

In SIAM J. Numer. Anal. 28 (1991) 1680–1697, Franca and Stenberg developed several Galerkin least squares methods for the solution of the problem of linear elasticity. That work concerned itself only with the error estimates of the method. It did not address the related problem of finding effective methods for the solution of the associated linear systems. In this work, we prove the convergence of a multigrid method. This multigrid is robust in that the convergence is uniform as the parameter ν goes to $1/2$. Computational experiments are included.

Keywords: Multigrid method; Linear elasticity; Galerkin least squares method; Stabilized method; Mixed finite element

1. Introduction

Let Ω be a bounded convex polygonal domain in R^2 and $\partial\Omega$ be the boundary of Ω. The pure displacement boundary value problem for planar linear elasticity is given in the form

$$\begin{aligned} &2\mu\left\{\underset{\sim}{\nabla}\cdot\underset{\approx}{\varepsilon}(\underset{\sim}{u})+\frac{\nu}{1-2\nu}\underset{\sim}{\nabla}\,\nabla\cdot\underset{\sim}{u}\right\}+\underset{\sim}{f}=\underset{\sim}{0}\quad \text{in } \Omega,\\ &\underset{\sim}{u}=\underset{\sim}{0}\quad \text{on } \partial\Omega. \end{aligned}\tag{1}$$

Here $\underset{\sim}{u}=(u_1,u_2)$ denotes the displacement, $\underset{\sim}{f}=(f_1,f_2)$ is the body force, ν is Poisson's ratio and μ is the shear modulus given by $\mu=E/\{2(1+\nu)\}$, where E is the Young's

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modulus. Instead of using Poisson's ratio ν and Young's elasticity modulus E, we can also work with the Lame constants λ and μ. These constants are related to each other by the following equations:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \nu = \frac{\lambda}{2(\lambda+\mu)},$$
$$\mu = \frac{E}{2(1+\nu)}, \qquad E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}.$$

We restrict Poisson's ratio to $0 \leqslant \nu < 1/2$ where the upper limit corresponds to an incompressible material.

Throughout this paper, we use a positive constant C independent of ν, mesh parameter h_k and grid level k which may vary from occurrence to occurrence even in the proof of the same theorem. We use undertildes to denote vector-valued functions, operators and their associated spaces, and double undertildes are used for matrix-valued functions and operators.

We use the following standard differential operators defined in [2,6]:

$$\nabla \cdot \underset{\sim}{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y},$$
$$\underset{\sim}{\nabla} \cdot \underset{\approx}{\tau} = \begin{pmatrix} \partial\tau_{11}/\partial x + \partial\tau_{12}/\partial y \\ \partial\tau_{21}/\partial x + \partial\tau_{22}/\partial y \end{pmatrix}, \qquad \underset{\approx}{\nabla}\, \underset{\sim}{v} = \begin{pmatrix} \partial v_1/\partial x & \partial v_1/\partial y \\ \partial v_2/\partial x & \partial v_2/\partial y \end{pmatrix},$$
$$\underset{\approx}{\tau} : \underset{\approx}{\eta} = \sum_{i=1}^{2}\sum_{j=1}^{2} \tau_{ij}\eta_{ij}, \quad \text{and} \quad \underset{\approx}{\varepsilon}(\underset{\sim}{v}) = \frac{1}{2}\big[\underset{\approx}{\nabla}\, \underset{\sim}{v} + (\underset{\approx}{\nabla}\, \underset{\sim}{v})^t\big].$$

Let $\underset{\sim}{H}^m(\Omega)$ denote the usual Sobolev space of functions with $L^2(\Omega)$ derivatives up to order m; see [5,6]. $\underset{\sim}{H}^m(\Omega)$ is equipped with the norm

$$\|\underset{\sim}{v}\|_{\underset{\sim}{H}^m(\Omega)} := \left(\int_\Omega \sum_{|\alpha|\leqslant m} |\partial^\alpha \underset{\sim}{v}|^2 \, dx\, dy \right)^{1/2}.$$

We use the following convention for the Sobolev seminorms:

$$|\underset{\sim}{v}|_{\underset{\sim}{H}^m(\Omega)} := \left(\int_\Omega \sum_{|\alpha| = m} |\partial^\alpha \underset{\sim}{v}|^2 \, dx\, dy \right)^{1/2}.$$

Let $\underset{\sim}{H}_0^m(\Omega) = \{\underset{\sim}{v} \in \underset{\sim}{H}^m(\Omega)\colon\ \underset{\sim}{v}|_{\partial\Omega} = 0\}$.

It is well known that for $\underset{\sim}{f} \in \underset{\sim}{L}^2(\Omega)$, Eq. (1) has a unique solution $\underset{\sim}{u} \in \underset{\sim}{H}^2(\Omega) \cap \underset{\sim}{H}_0^1(\Omega)$; see [9].

It is well known that one way of stabilizing mixed finite element methods is to combine the classical Galerkin formulation with least-squares forms of the differential equations (see [4,5,7,8]). An advantage of this method is that the class of finite element spaces that can be used is considerably enlarged, hence the methods are easily incorporated into existing finite element codes. In Theorem 1, we need the stabilization parameter α which is bounded by C_I, where the constant C_I is related to the inverse inequality and a measure of the regularity of the triangulation. But C_I is unknown. So, for the implementation of stabilized mixed finite element methods, we have to analyze the behaviors of α and C_I in order to obtain rapid iterative convergence.

As documented in [11], the standard multigrid method using conforming bilinear finite elements requires a large number of smoothing steps in order to achieve convergence for nearly incompressible linear elasticity problems. Our algorithm converges with a small number of smoothing steps, but we need a large number of iterations; see [2,10]. Also, we can reduce the number of iterations in our algorithm by taking many smoothing steps. We use $P-1$ finite element spaces for approximating both the displacement and the pressure in the implementation.

There are many papers regarding the convergence proof for a W-cycle multigrid method of the mixed problem; see [1,2,10,12]. For example, Brenner [2] provides a convergence proof for a W-cycle multigrid method for a nonconforming method in linear elasticity. Also, Lee [10] gives a similar convergence proof for the pure traction problem. In this paper, we use the techniques of Brenner and Lee to obtain analogous results to them, but applied to Franca and Stenberg's stabilized formulation. The main difficulties in the analysis is to provide appropriate bounds on the least squares terms. We prove the convergence of a W-cycle multigrid method and show that the convergence is uniform with respect to the parameter ν. Moreover, we show that the number of iterations for the W-cycle multigrid methods is reduced by a half when we take twice as many smoothing steps in the algorithm and also reduced by a half when we cut the mesh size by a half. Brenner [2] reports very similar results for the pure displacement boundary value problem with the nonconforming finite element method.

This paper is organized as follows. We explain the conforming finite element method in Section 2. We discuss the multigrid algorithm in Section 3. We prove the convergence of a W-cycle multigrid method in Section 4. The computational results are presented in Section 5.

2. The finite element method

For simplicity, we assume that $2\mu = 1$. Let $p = -(1/\epsilon)\nabla \cdot \underset{\sim}{u}$, where $\epsilon = (1-2\nu)/\nu$. Then (1) is equivalent to

$$\begin{aligned}
&-\underset{\sim}{\nabla} \cdot \underset{\approx}{\varepsilon}(\underset{\sim}{u}) + \underset{\sim}{\nabla}\, p = \underset{\sim}{f} \quad \text{in } \Omega,\\
&\epsilon p + \nabla \cdot \underset{\sim}{u} = 0 \quad \text{in } \Omega,\\
&\underset{\sim}{u} = \underset{\sim}{0} \quad \text{on } \partial\Omega.
\end{aligned} \tag{2}$$

Hence, we have the following weak formulation: Find $(\underset{\sim}{u}, p) \in \underset{\sim}{H}_0^1(\Omega) \times L^2(\Omega)$ such that

$$\int_\Omega \underset{\approx}{\varepsilon}(\underset{\sim}{u}) : \underset{\approx}{\varepsilon}(\underset{\sim}{v})\, dx\, dy - \int_\Omega (\nabla \cdot \underset{\sim}{v}) p\, dx\, dy = \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}\, dx\, dy, \quad \forall \underset{\sim}{v} \in \underset{\sim}{H}_0^1(\Omega),$$

$$\epsilon \int_{\Omega} pq\,dx\,dy + \int_{\Omega} (\nabla \cdot \underset{\sim}{u})q\,dx\,dy = 0, \quad \forall q \in L^2(\Omega). \tag{3}$$

Let $\mathcal{T}^k$ be a family of triangulations of Ω, where $\mathcal{T}^{k+1}$ be obtained by connecting the midpoints of the edges of the triangles in $\mathcal{T}^k$. Let $h_T = \operatorname{diam}(T)$ for each $T \in \mathcal{T}^k$ and $h_k = \max_{T\in\mathcal{T}^k} h_T$; then $h_k = 2h_{k+1}$. Now let us define the conforming finite element spaces for our multigrid method.

$$\underset{\sim}{V}_k := \left\{ \underset{\sim}{v} \in \underset{\sim}{C}^0(\Omega);\ \underset{\sim}{v}|_T \text{ is linear for all } T \in \mathcal{T}^k \text{ and } \underset{\sim}{v}\,|_{\partial\Omega} = \underset{\sim}{0} \right\}$$

and

$$P_k := \left\{ q \in C^0(\Omega);\ q|_T \text{ is linear for all } T \in \mathcal{T}^k \right\}.$$

Then the discretized problem for (3) is the following: Find $(\underset{\sim}{u}_k, p_k) \in \underset{\sim}{V}_k \times P_k$ such that

$$\mathcal{B}_k\big((\underset{\sim}{u}_k, p_k), (\underset{\sim}{v}_k, q_k)\big) = \mathcal{F}_{\underset{\sim}{f}}(\underset{\sim}{v}_k, q_k), \quad \forall (\underset{\sim}{v}_k, q_k) \in \underset{\sim}{V}_k \times P_k, \tag{4}$$

where

$$\begin{aligned}
&\mathcal{B}_k\big((\underset{\sim}{u}_k, p_k), (\underset{\sim}{v}_k, q_k)\big)\\
&\quad = \int_{\Omega} \underset{\approx}{\varepsilon}(\underset{\sim}{u}_k) : \underset{\approx}{\varepsilon}(\underset{\sim}{v}_k)\,dx\,dy - \int_{\Omega} (\nabla \cdot \underset{\sim}{u}_k) q_k\,dx\,dy - \int_{\Omega} (\nabla \cdot \underset{\sim}{v}_k) p_k\,dx\,dy\\
&\qquad - \alpha \sum_{T\in\mathcal{T}^k} h_T^2 \int_T \big(-\underset{\sim}{\nabla} \cdot \underset{\approx}{\varepsilon}(\underset{\sim}{u}_k) + \underset{\sim}{\nabla}\, p_k\big) \cdot \big(-\underset{\sim}{\nabla} \cdot \underset{\approx}{\varepsilon}(\underset{\sim}{v}_k) + \underset{\sim}{\nabla}\, q_k\big)\,dx\,dy\\
&\qquad - \epsilon \int_{\Omega} p_k q_k\,dx\,dy
\end{aligned}$$

and

$$\mathcal{F}_{\underset{\sim}{f}}(\underset{\sim}{v}_k, q_k) = \int_{\Omega} \underset{\sim}{f} \cdot \underset{\sim}{v}_k\,dx\,dy - \alpha \sum_{T\in\mathcal{T}^k} h_T^2 \int_T \underset{\sim}{f} \cdot \big(-\underset{\sim}{\nabla} \cdot \underset{\approx}{\varepsilon}(\underset{\sim}{v}_k) + \underset{\sim}{\nabla}\, q_k\big)\,dx\,dy.$$

Note that the bilinear form $\mathcal{B}_k$ is symmetric and indefinite, but the equation for the displacement becomes positive definite once the pressure is eliminated.

In [8], Franca and Stenberg proved the uniqueness of the solution of the conforming discretization (4) and derived the following discretization error estimate.

Theorem 1. *Let $(\underset{\sim}{u}, p)$ be the solution of (2). Then for $0 < \alpha < C_I$, (4) has a unique solution satisfying*

$$\|\underset{\sim}{u} - \underset{\sim}{u}_k\|_{\underset{\sim}{H}^1(\Omega)} + \|p - p_k\|_{L^2(\Omega)} \leqslant C h_k \|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)}.$$

In addition, by the classical Aubin–Nitsche trick, we have

$$\|\underset{\sim}{u} - \underset{\sim}{u}_k\|_{\underset{\sim}{L}^2(\Omega)} \leqslant C {h_k}^2 \|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)}.$$

The constant C_I is the largest value satisfying the inverse inequality

$$C_I \sum_{T\in\mathcal{T}^k} h_T^2 \left\| \underset{\sim}{\nabla} \cdot \underset{\approx}{\varepsilon}(\underset{\sim}{v}_k)\right\|^2_{\underset{\sim}{L}^2(T)} \leqslant \left\|\underset{\approx}{\varepsilon}(\underset{\sim}{v}_k)\right\|^2_{\underset{\approx}{L}^2(\Omega)}, \quad \forall \underset{\sim}{v}_k \in \underset{\sim}{V}_k.$$

3. Multigrid algorithm

In this section, we define the intergrid transfer operators and the mesh dependent norms and present some basic lemmas which are used to prove the convergence of the algorithm. Some of them are rewordings of lemmas in [2,10], and we give these lemmas without proof.

In order to define the fine-to-coarse operator I_k^{k-1}, we introduce the following mesh-dependent inner product:

$$\big((\underset{\sim}{u}, p), (\underset{\sim}{v}, q)\big)_k := (\underset{\sim}{u}, \underset{\sim}{v})_{\underset{\sim}{L}^2(\Omega)} + h_k^2 (p, q)_{L^2(\Omega)}.$$

Then $I_k^{k-1} : \underset{\sim}{V}_k \times P_k \to \underset{\sim}{V}_{k-1} \times P_{k-1}$ is defined by

$$\big(I_k^{k-1}(\underset{\sim}{u}, p), (\underset{\sim}{v}, q)\big)_{k-1} = \big((\underset{\sim}{u}, p), (\underset{\sim}{v}, q)\big)_k$$

for all $(\underset{\sim}{u}, p) \in \underset{\sim}{V}_k \times P_k$ and $(\underset{\sim}{v}, q) \in \underset{\sim}{V}_{k-1} \times P_{k-1}$.

Define $B_k : \underset{\sim}{V}_k \times P_k \to \underset{\sim}{V}_k \times P_k$ by

$$\big(B_k(\underset{\sim}{u}, p), (\underset{\sim}{v}, q)\big)_k = \mathcal{B}_k\big((\underset{\sim}{u}, p), (\underset{\sim}{v}, q)\big)$$

for all $(\underset{\sim}{u}, p), (\underset{\sim}{v}, q) \in \underset{\sim}{V}_k \times P_k$.

The mesh and parameters (α and ϵ)-dependent norms on $\underset{\sim}{V}_k \times P_k$ are defined as follows:

$$\left|\!\left|\!\left|(\underset{\sim}{u}, p)\right|\!\right|\!\right|_{s,k} := \sqrt{\big((B_k^2)^{s/2}(\underset{\sim}{u}, p), (\underset{\sim}{u}, p)\big)_k} \quad \text{for all } (\underset{\sim}{u}, p) \in \underset{\sim}{V}_k \times P_k.$$

Note that B_k is nonsingular and symmetric, hence ${B_k}^2$ is positive definite with respect to $(\cdot\,,\cdot)_k$. Therefore, this norm is well defined for each $s \in R$. Moreover,

$$\left|\!\left|\!\left|(\underset{\sim}{u}, p)\right|\!\right|\!\right|_{0,k} := \sqrt{\|\underset{\sim}{u}\|^2_{\underset{\sim}{L}^2(\Omega)} + h_k^2 \|p\|^2_{L^2(\Omega)}} \quad \text{for all } (\underset{\sim}{u}, p) \in \underset{\sim}{V}_k \times P_k,$$

$$\left|\mathcal{B}_k\big((\underset{\sim}{u}, p), (\underset{\sim}{v}, q)\big)\right| \leqslant \left|\!\left|\!\left|(\underset{\sim}{u}, p)\right|\!\right|\!\right|_{2,k} \left|\!\left|\!\left|(\underset{\sim}{v}, q)\right|\!\right|\!\right|_{0,k} \quad \text{for all } (\underset{\sim}{u}, p), (\underset{\sim}{v}, q) \in \underset{\sim}{V}_k \times P_k,$$

and

$$\left|\!\left|\!\left|(\underset{\sim}{u}, p)\right|\!\right|\!\right|_{2,k} = \sup_{(\underset{\sim}{v}, q)\in \underset{\sim}{V}_k\times P_k\setminus\{(\underset{\sim}{0},0)\}} \frac{|\mathcal{B}_k((\underset{\sim}{u}, p), (\underset{\sim}{v}, q))|}{|\!|\!|(\underset{\sim}{v}, q)|\!|\!|_{0,k}} \quad \text{for all } (\underset{\sim}{u}, p) \in \underset{\sim}{V}_k \times P_k.$$

Let

$$\bar{\mathcal{B}}_{k-1}\big((\underset{\sim}{u}, p), (\underset{\sim}{v}, q)\big) = \int_\Omega \underset{\approx}{\varepsilon}(\underset{\sim}{u}) : \underset{\approx}{\varepsilon}(\underset{\sim}{v})\,dx\,dy - \int_\Omega (\nabla\cdot\underset{\sim}{u})q\,dx\,dy - \int_\Omega (\nabla\cdot\underset{\sim}{v})p\,dx\,dy - \frac{\alpha}{4}\sum_{T\in\mathcal{T}^{k-1}} h_T^2\big(-\underset{\sim}{\nabla}\cdot\underset{\approx}{\varepsilon}(\underset{\sim}{u}) + \underset{\sim}{\nabla}\,p, -\underset{\sim}{\nabla}\cdot\underset{\approx}{\varepsilon}(\underset{\sim}{v}) + \underset{\sim}{\nabla}\,q\big)_{\underset{\sim}{L}^2(T)} - \epsilon\int_\Omega pq\,dx\,dy$$

and

$$\bar{\mathcal{F}}_{\underset{\sim}{f}}(\underset{\sim}{v}, q) = \int_\Omega \underset{\sim}{f}\cdot\underset{\sim}{v}\,dx\,dy - \frac{\alpha}{4}\sum_{T\in\mathcal{T}^{k-1}} h_T^2\big(\underset{\sim}{f}, -\underset{\sim}{\nabla}\cdot\underset{\approx}{\varepsilon}(\underset{\sim}{v}) + \underset{\sim}{\nabla}\,q\big)_{\underset{\sim}{L}^2(T)}.$$

Note that

$$\bar{\mathcal{B}}_{k-1} \quad\text{and}\quad \bar{\mathcal{F}}_{\underset{\sim}{f}}$$

are different from

$$\mathcal{B}_{k-1} \quad\text{and}\quad \mathcal{F}_{\underset{\sim}{f}}.$$

The difference is in the least squares term. We divide the stabilization parameter α by 4 to define $\bar{\mathcal{B}}_{k-1}$ and $\bar{\mathcal{F}}_{\underset{\sim}{f}}$.

Define $P_k^{k-1} : \underset{\sim}{V}_k \times P_k \to \underset{\sim}{V}_{k-1} \times P_{k-1}$ by

$$\bar{\mathcal{B}}_{k-1}\big(P_k^{k-1}(\underset{\sim}{u}, p), (\underset{\sim}{v}, q)\big) = \mathcal{B}_k\big((\underset{\sim}{u}, p), (\underset{\sim}{v}, q)\big)$$

for all $(\underset{\sim}{u}, p) \in \underset{\sim}{V}_k \times P_k$ and $(\underset{\sim}{v}, q) \in \underset{\sim}{V}_{k-1} \times P_{k-1}$.

The proofs of the following lemmas are straightforward by using the techniques of [2,10] once we provide appropriate bounds on the extra sum of least squares terms. Therefore, it is enough to estimate the least squares terms. To estimate the extra sum of the least squares terms, we use the inverse inequality, the regularity of the mesh and Lemma 3.2 in [8]. Then, for $\underset{\sim}{u}_k \in \underset{\sim}{V}_k$, $\underset{\sim}{u}_{k-1} \in \underset{\sim}{V}_{k-1}$ and $p_k \in P_k$, we have

$$\begin{aligned}
&\Big|\alpha\sum_{T\in\mathcal{T}^k} h_T^2\big(\underset{\sim}{u}_k - \underset{\sim}{u}_{k-1}, -\underset{\sim}{\nabla}\cdot\underset{\approx}{\varepsilon}(\underset{\sim}{u}_k) + \underset{\sim}{\nabla}\,p_k\big)_{\underset{\sim}{L}^2(T)}\Big| \\
&\quad\leqslant C\alpha h_k\|\underset{\sim}{u}_k - \underset{\sim}{u}_{k-1}\|_{\underset{\sim}{L}^2(\Omega)}\|\underset{\sim}{u}_k\|_{\underset{\sim}{H}^1(\Omega)} + C\alpha h_k\|\underset{\sim}{u}_k - \underset{\sim}{u}_{k-1}\|_{\underset{\sim}{L}^2(\Omega)}\|p_k\|_{L^2(\Omega)} \\
&\quad\leqslant C\sqrt{2}\,\alpha h_k\|\underset{\sim}{u}_k - \underset{\sim}{u}_{k-1}\|_{\underset{\sim}{L}^2(\Omega)}\big(\|\underset{\sim}{u}_k\|^2_{\underset{\sim}{H}^1(\Omega)} + \|p_k\|^2_{L^2(\Omega)}\big)^{1/2} \\
&\quad\leqslant C\sqrt{2}\,\alpha h_k\|\underset{\sim}{u}_k - \underset{\sim}{u}_{k-1}\|_{\underset{\sim}{L}^2(\Omega)} \\
&\qquad\times\sup_{(\underset{\sim}{v},q)\in\underset{\sim}{V}_k\times P_k\setminus\{(\underset{\sim}{0},0)\}}\frac{|\mathcal{B}_k((\underset{\sim}{u}_k, p_k), (\underset{\sim}{v}, q))|}{(\|\underset{\sim}{v}\|^2_{\underset{\sim}{H}^1(\Omega)} + (1+\epsilon)\|q\|^2_{L^2(\Omega)})^{1/2}}.
\end{aligned}\tag{5}$$

We present some basic lemmas which are essential in the proof of the approximation property (Lemma 4) of the conforming multigrid algorithm.

Lemma 1. *Given* $\underset{\sim}{\omega} \in \underset{\sim}{L}^2(\Omega)$, *let* $(\underset{\sim}{u}_k, p_k) \in \underset{\sim}{V}_k \times P_k$ *be the solution of*

$$\mathcal{B}_k\big((\underset{\sim}{u}_k, p_k), (\underset{\sim}{v}, q)\big) = \mathcal{F}_{\underset{\sim}{\omega}}(\underset{\sim}{v}, q), \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V}_k \times P_k,$$

and let $(\underset{\sim}{u}_{k-1}, p_{k-1}) \in \underset{\sim}{V}_{k-1} \times P_{k-1}$ *be the solution of*

$$\bar{\mathcal{B}}_{k-1}\big((\underset{\sim}{u}_{k-1}, p_{k-1}), (\underset{\sim}{v}, q)\big) = \bar{\mathcal{F}}_{\underset{\sim}{\omega}}(\underset{\sim}{v}, q), \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V}_{k-1} \times P_{k-1}.$$

Then $|||(\underset{\sim}{u}_k, p_k) - (\underset{\sim}{u}_{k-1}, p_{k-1})|||_{0,k} \leqslant C h_k^2 \|\underset{\sim}{\omega}\|_{\underset{\sim}{L}^2(\Omega)}$.

Proof. See the proof of Lemma 4 in [10] and Theorem 1. □

Lemma 2. *Given* $\omega \in L^2(\Omega)$, *let* $(\underset{\sim}{u}_k, p_k) \in \underset{\sim}{V}_k \times P_k$ *be the solution of*

$$\mathcal{B}_k\big((\underset{\sim}{u}_k, p_k), (\underset{\sim}{v}, q)\big) = \int_\Omega wq\, dx\, dy, \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V}_k \times P_k,$$

and let $(\underset{\sim}{u}_{k-1}, p_{k-1}) \in \underset{\sim}{V}_{k-1} \times P_{k-1}$ *be the solution of*

$$\bar{\mathcal{B}}_{k-1}\big((\underset{\sim}{u}_{k-1}, p_{k-1}), (\underset{\sim}{v}, q)\big) = \int_\Omega wq\, dx\, dy, \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V}_{k-1} \times P_{k-1}.$$

Then $|||(\underset{\sim}{u}_k, p_k) - (\underset{\sim}{u}_{k-1}, p_{k-1})|||_{0,k} \leqslant C h_k \|\omega\|_{L^2(\Omega)}$.

Proof. By the definition of $(\underset{\sim}{u}_k, p_k)$ and (5), we have

$$\begin{aligned}
&\left|\alpha \sum_{T \in \mathcal{T}^k} h_T^2 \big(\underset{\sim}{u}_k - \underset{\sim}{u}_{k-1}, -\underset{\sim}{\nabla} \cdot \underset{\approx}{\varepsilon}(\underset{\sim}{u}_k) + \underset{\sim}{\nabla}\, p_k\big)_{\underset{\sim}{L}^2(T)}\right| \\
&\quad \leqslant C\sqrt{2}\, \alpha h_k \|\underset{\sim}{u}_k - \underset{\sim}{u}_{k-1}\|_{\underset{\sim}{L}^2(\Omega)} \\
&\qquad \times \sup_{(\underset{\sim}{v}, q) \in \underset{\sim}{V}_k \times P_k \setminus \{(\underset{\sim}{0}, 0)\}} \frac{|\mathcal{B}_k((\underset{\sim}{u}_k, p_k), (\underset{\sim}{v}, q))|}{(\|\underset{\sim}{v}\|^2_{\underset{\sim}{H}^1(\Omega)} + (1+\epsilon)\|q\|^2_{L^2(\Omega)})^{1/2}} \\
&\quad \leqslant C\sqrt{2}\, \alpha h_k \|\underset{\sim}{u}_k - \underset{\sim}{u}_{k-1}\|_{\underset{\sim}{L}^2(\Omega)} \\
&\qquad \times \sup_{(\underset{\sim}{v}, q) \in \underset{\sim}{V}_k \times P_k \setminus \{(\underset{\sim}{0}, 0)\}} \frac{\|\omega\|_{L^2(\Omega)} \|q\|_{L^2(\Omega)}}{(\|\underset{\sim}{v}\|^2_{\underset{\sim}{H}^1(\Omega)} + (1+\epsilon)\|q\|^2_{L^2(\Omega)})^{1/2}} \\
&\quad \leqslant C h_k \|\underset{\sim}{u}_k - \underset{\sim}{u}_{k-1}\|_{\underset{\sim}{L}^2(\Omega)} \|\omega\|_{L^2(\Omega)} \quad (\text{take } C := C\sqrt{2}\alpha).
\end{aligned}$$

The remaining of the proof is very similar to the proof of Lemma 5 in [10] with the help of the above estimate. □

4. Convergence analysis

In this section, we provide a convergence analysis of the W-cycle multigrid algorithm because the analysis of the W-cycle is much simpler than the analysis of the V-cycle, being able to be obtained via a simple perturbation argument based on the two grid analysis.

In [2,10], the kth level and two grid iteration scheme of the W-cycle multigrid algorithm are well introduced. We use the same notation for the relaxation operator R_k, the intergrid transfer operator P_k^{k-1} and I_{k-1}^k as in [2,3,10]. The proof of the following smoothing property is standard, so it is omitted; see [3].

Lemma 3 (Smoothing step). *There exists a constant C, independent of h_k and m, such that*

$$\left\|\left|R_k^m(\underset{\sim}{u}, p)\right\|\right|_{2,k} \leqslant C h_k^{-2} \frac{1}{\sqrt{m}} \left\|\left|(\underset{\sim}{u}, p)\right\|\right|_{0,k}, \quad \forall(\underset{\sim}{u}, p) \in \underset{\sim}{V}_k \times P_k.$$

The proof of the following lemma is very similar to the method in [2,10] except the estimation of the extra sum of the least squares term. With the definition of $\bar{\mathcal{B}}_{k-1}$ and $\bar{\mathcal{F}}_f$, and appropriate bounds on the least squares term, we have the following lemma.

Lemma 4 (Approximation step). *There exists a constant C, independent of h_k and m, such that*

$$\left\|\left|\left(I - P_k^{k-1}\right)(\underset{\sim}{u}, p)\right\|\right|_{0,k} \leqslant C h_k^2 \left\|\left|(\underset{\sim}{u}, p)\right\|\right|_{2,k}, \quad \forall(\underset{\sim}{u}, p) \in \underset{\sim}{V}_k \times P_k.$$

Proof. Let

$$(\underset{\sim}{\eta}, \tau) = P_k^{k-1}(\underset{\sim}{u}, p) \quad \text{for any } (\underset{\sim}{u}, p) \in \underset{\sim}{V}_k \times P_k.$$

Then

$$\left(I - P_k^{k-1}\right)(\underset{\sim}{u}, p) = (\underset{\sim}{u} - \underset{\sim}{\eta}, p - \tau)$$

and

$$\left\|\left|(\underset{\sim}{u} - \underset{\sim}{\eta}, p - \tau)\right\|\right|_{0,k}^2 = \| \underset{\sim}{u} - \underset{\sim}{\eta}\|_{L^2(\Omega)}^2 + h_k^2 \|p - \tau\|_{L^2(\Omega)}^2.$$

First, we will estimate $\|p - \tau\|_{L^2(\Omega)}$ by a duality argument. Let $(\underset{\sim}{\varphi}_k, \psi_k) \in \underset{\sim}{V}_k \times P_k$ be the solution of

$$\mathcal{B}_k\left((\underset{\sim}{\varphi}_k, \psi_k), (\underset{\sim}{v}, q)\right) = \int_\Omega (p - \tau) q \, dx \, dy, \quad \forall(\underset{\sim}{v}, q) \in \underset{\sim}{V}_k \times P_k,$$

and $(\underset{\sim}{\varphi}_{k-1}, \psi_{k-1}) \in \underset{\sim}{V}_{k-1} \times P_{k-1}$ be the solution of

$$\bar{\mathcal{B}}_{k-1}\left((\underset{\sim}{\varphi}_{k-1}, \psi_{k-1}), (\underset{\sim}{v}, q)\right) = \int_\Omega (p - \tau) q \, dx \, dy, \quad \forall(\underset{\sim}{v}, q) \in \underset{\sim}{V}_{k-1} \times P_{k-1}.$$

Then

$$\begin{aligned}
\|p-\tau\|^2_{L^2(\Omega)} &= \mathcal{B}_k\big((\underset{\sim}{\varphi}_k,\psi_k),(\underset{\sim}{u},p)\big)-\bar{\mathcal{B}}_{k-1}\big((\underset{\sim}{\varphi}_{k-1},\psi_{k-1}),(\underset{\sim}{\eta},\tau)\big)\\
&= \mathcal{B}_k\big((\underset{\sim}{\varphi}_k,\psi_k),(\underset{\sim}{u},p)\big)-\bar{\mathcal{B}}_{k-1}\big((\underset{\sim}{\varphi}_{k-1},\psi_{k-1}),P_k^{k-1}(\underset{\sim}{u},p)\big)\\
&= \mathcal{B}_k\big((\underset{\sim}{\varphi}_k,\psi_k),(\underset{\sim}{u},p)\big)-\mathcal{B}_k\big((\underset{\sim}{\varphi}_{k-1},\psi_{k-1}),(\underset{\sim}{u},p)\big)\\
&= \mathcal{B}_k\big((\underset{\sim}{\varphi}_k,\psi_k)-(\underset{\sim}{\varphi}_{k-1},\psi_{k-1}),(\underset{\sim}{u},p)\big)\\
&\leqslant \big|\!\big|\!\big|(\underset{\sim}{\varphi}_k,\psi_k)-(\underset{\sim}{\varphi}_{k-1},\psi_{k-1})\big|\!\big|\!\big|_{0,k}\big|\!\big|\!\big|(\underset{\sim}{u},p)\big|\!\big|\!\big|_{2,k}\\
&\leqslant Ch_k\|p-\tau\|_{L^2(\Omega)}\big|\!\big|\!\big|(\underset{\sim}{u},p)\big|\!\big|\!\big|_{2,k} \quad \text{(by Lemma 2).}
\end{aligned}$$

Therefore,

$$\|p-\tau\|_{L^2(\Omega)} \leqslant Ch_k\big|\!\big|\!\big|(\underset{\sim}{u},p)\big|\!\big|\!\big|_{2,k}. \tag{6}$$

Next, we want to estimate $\|\underset{\sim}{u}-\underset{\sim}{\eta}\|_{\underset{\sim}{L}^2(\Omega)}$. Let $(\underset{\sim}{\zeta}_k,\xi_k)\in \underset{\sim}{V}_k\times P_k$ be the solution of

$$\mathcal{B}_k\big((\underset{\sim}{\zeta}_k,\xi_k),(\underset{\sim}{v},q)\big)=\mathcal{F}_{\underset{\sim}{u}-\underset{\sim}{\eta}}(\underset{\sim}{v},q), \quad \forall(\underset{\sim}{v},q)\in\underset{\sim}{V}_k\times P_k,$$

and $(\underset{\sim}{\zeta}_{k-1},\xi_{k-1})\in\underset{\sim}{V}_{k-1}\times P_{k-1}$ be the solution of

$$\bar{\mathcal{B}}_{k-1}\big((\underset{\sim}{\zeta}_{k-1},\xi_{k-1}),(\underset{\sim}{v},q)\big)=\bar{\mathcal{F}}_{\underset{\sim}{u}-\underset{\sim}{\eta}}(\underset{\sim}{v},q), \quad \forall(\underset{\sim}{v},q)\in\underset{\sim}{V}_{k-1}\times P_{k-1}.$$

Then

$$\begin{aligned}
&\mathcal{B}_k\big((\underset{\sim}{\zeta}_k,\xi_k),(\underset{\sim}{u},p)\big)-\bar{\mathcal{B}}_{k-1}\big((\underset{\sim}{\zeta}_{k-1},\xi_{k-1}),(\underset{\sim}{\eta},\tau)\big)\\
&\quad=\|\underset{\sim}{u}-\underset{\sim}{\eta}\|^2_{\underset{\sim}{L}^2(\Omega)}-\alpha\sum_{T\in\mathcal{T}^k}h_T^2\big(\underset{\sim}{u}-\underset{\sim}{\eta},-\underset{\sim}{\nabla}\cdot\underset{\approx}{\varepsilon}(\underset{\sim}{u})+\underset{\sim}{\nabla}p\big)_{\underset{\sim}{L}^2(T)}\\
&\qquad+\frac{\alpha}{4}\sum_{T\in\mathcal{T}^{k-1}}h_T^2\big(\underset{\sim}{u}-\underset{\sim}{\eta},-\underset{\sim}{\nabla}\cdot\underset{\approx}{\varepsilon}(\underset{\sim}{\eta})+\underset{\sim}{\nabla}\tau\big)_{\underset{\sim}{L}^2(T)}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\|\underset{\sim}{u}-\underset{\sim}{\eta}\|^2_{\underset{\sim}{L}^2(\Omega)} &= \mathcal{B}_k\big((\underset{\sim}{\zeta}_k,\xi_k),(\underset{\sim}{u},p)\big)-\bar{\mathcal{B}}_{k-1}\big((\underset{\sim}{\zeta}_{k-1},\xi_{k-1}),(\underset{\sim}{\eta},\tau)\big)\\
&\quad+\alpha\sum_{T\in\mathcal{T}^k}h_T^2\big(\underset{\sim}{u}-\underset{\sim}{\eta},-\underset{\sim}{\nabla}\cdot\underset{\approx}{\varepsilon}(\underset{\sim}{u})+\underset{\sim}{\nabla}p\big)_{\underset{\sim}{L}^2(T)}\\
&\quad-\frac{\alpha}{4}\sum_{T\in\mathcal{T}^{k-1}}h_T^2\big(\underset{\sim}{u}-\underset{\sim}{\eta},-\underset{\sim}{\nabla}\cdot\underset{\approx}{\varepsilon}(\underset{\sim}{\eta})+\underset{\sim}{\nabla}\tau\big)_{\underset{\sim}{L}^2(T)}\\
&= \mathcal{B}_k\big((\underset{\sim}{\zeta}_k,\xi_k),(\underset{\sim}{u},p)\big)-\bar{\mathcal{B}}_{k-1}\big((\underset{\sim}{\zeta}_{k-1},\xi_{k-1}),(\underset{\sim}{\eta},\tau)\big)\\
&\quad+\alpha\sum_{T\in\mathcal{T}^k}h_T^2\big(\underset{\sim}{u}-\underset{\sim}{\eta},-\underset{\sim}{\nabla}\cdot\underset{\approx}{\varepsilon}(\underset{\sim}{u}-\underset{\sim}{\eta})+\underset{\sim}{\nabla}(p-\tau)\big)_{\underset{\sim}{L}^2(T)}.
\end{aligned} \tag{7}$$

Now we want to estimate

$$\left|\alpha \sum_{T\in\mathcal{T}^k} h_T^2 \big(\underset{\sim}{u}-\underset{\sim}{\eta}, -\underset{\sim}{\nabla}\cdot \underset{\approx}{\varepsilon}(\underset{\sim}{u}-\underset{\sim}{\eta}) + \underset{\sim}{\nabla}(p-\tau)\big)_{\underset{\sim}{L}^2(T)}\right|.$$

By the inverse inequality and the regularity of the mesh, it is easy to show that

$$\begin{aligned}
&\left|\alpha \sum_{T\in\mathcal{T}^k} h_T^2 \big(\underset{\sim}{u}-\underset{\sim}{\eta}, -\underset{\sim}{\nabla}\cdot \underset{\approx}{\varepsilon}(\underset{\sim}{u}-\underset{\sim}{\eta}) + \underset{\sim}{\nabla}(p-\tau)\big)_{\underset{\sim}{L}^2(T)}\right| \\
&\quad\leqslant C\alpha h_k \|\underset{\sim}{u}-\underset{\sim}{\eta}\|_{\underset{\sim}{L}^2(\Omega)} \|\underset{\sim}{u}-\underset{\sim}{\eta}\|_{\underset{\sim}{H}^1(\Omega)} + C\alpha h_k \|\underset{\sim}{u}-\underset{\sim}{\eta}\|_{\underset{\sim}{L}^2(\Omega)} \|p-\tau\|_{L^2(\Omega)} \\
&\quad\leqslant C\sqrt{C_I}\sqrt{\alpha}\|\underset{\sim}{u}-\underset{\sim}{\eta}\|^2_{\underset{\sim}{L}^2(\Omega)} + CC_I h_k \|\underset{\sim}{u}-\underset{\sim}{\eta}\|_{\underset{\sim}{L}^2(\Omega)} \|p-\tau\|_{L^2(\Omega)} \quad \text{for } \alpha < C_I.
\end{aligned}$$

Therefore, for $\alpha < C_I$, (7) can be written as

$$\begin{aligned}
\|\underset{\sim}{u}-\underset{\sim}{\eta}\|^2_{\underset{\sim}{L}^2(\Omega)} &\leqslant \mathcal{B}_k\big((\underset{\sim}{\zeta}_k,\xi_k),(\underset{\sim}{u},p)\big) - \bar{\mathcal{B}}_{k-1}\big((\underset{\sim}{\zeta}_{k-1},\xi_{k-1}),(\underset{\sim}{\eta},\tau)\big) \\
&\qquad + C\sqrt{C_I}\sqrt{\alpha}\|\underset{\sim}{u}-\underset{\sim}{\eta}\|^2_{\underset{\sim}{L}^2(\Omega)} + CC_I h_k \|\underset{\sim}{u}-\underset{\sim}{\eta}\|_{\underset{\sim}{L}^2(\Omega)} \|p-\tau\|_{L^2(\Omega)} \\
&\leqslant \big|\!\big|\!\big|(\underset{\sim}{\zeta}_k,\xi_k)-(\underset{\sim}{\zeta}_{k-1},\xi_{k-1})\big|\!\big|\!\big|_{0,k} \big|\!\big|\!\big|(\underset{\sim}{u},p)\big|\!\big|\!\big|_{2,k} \\
&\qquad + C\sqrt{C_I}\sqrt{\alpha}\|\underset{\sim}{u}-\underset{\sim}{\eta}\|^2_{\underset{\sim}{L}^2(\Omega)} + CC_I h_k^2 \|\underset{\sim}{u}-\underset{\sim}{\eta}\|_{\underset{\sim}{L}^2(\Omega)} \big|\!\big|\!\big|(\underset{\sim}{u},p)\big|\!\big|\!\big|_{2,k}.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
&(1-C\sqrt{C_I}\sqrt{\alpha})\|\underset{\sim}{u}-\underset{\sim}{\eta}\|^2_{\underset{\sim}{L}^2(\Omega)} \\
&\quad\leqslant \big|\!\big|\!\big|(\underset{\sim}{\zeta}_k,\xi_k)-(\underset{\sim}{\zeta}_{k-1},\xi_{k-1})\big|\!\big|\!\big|_{0,k} \big|\!\big|\!\big|(\underset{\sim}{u},p)\big|\!\big|\!\big|_{2,k} + CC_I h_k^2 \|\underset{\sim}{u}-\underset{\sim}{\eta}\|_{\underset{\sim}{L}^2(\Omega)} \big|\!\big|\!\big|(\underset{\sim}{u},p)\big|\!\big|\!\big|_{2,k} \\
&\quad\leqslant Ch_k^2 \|\underset{\sim}{u}-\underset{\sim}{\eta}\|_{\underset{\sim}{L}^2(\Omega)} \big|\!\big|\!\big|(\underset{\sim}{u},p)\big|\!\big|\!\big|_{2,k} \quad \text{(by Lemma 1).}
\end{aligned}$$

Therefore, it follows that

$$\|\underset{\sim}{u}-\underset{\sim}{\eta}\|_{\underset{\sim}{L}^2(\Omega)} \leqslant Ch_k^2 \big|\!\big|\!\big|(\underset{\sim}{u},p)\big|\!\big|\!\big|_{2,k} \tag{8}$$

for sufficiently small α satisfying $1-C\sqrt{C_I}\sqrt{\alpha} > 0$. By (6) and (8), we have

$$\big|\!\big|\!\big|(\underset{\sim}{u}-\underset{\sim}{\eta}, p-\tau)\big|\!\big|\!\big|^2_{0,k} \leqslant Ch_k^4 \big|\!\big|\!\big|(\underset{\sim}{u},p)\big|\!\big|\!\big|^2_{2,k}.$$

Thus we have $|\!|\!|(I-P_k^{k-1})(\underset{\sim}{u},p)|\!|\!|_{0,k} \leqslant Ch_k^2 |\!|\!|(\underset{\sim}{u},p)|\!|\!|_{2,k}$. $\square$

The proofs of the following two theorems are standard.

Theorem 2 (Convergence of the two-grid algorithm). *There exists a constant C, independent of k and m, such that*

$$\big|\!\big|\!\big|(\underset{\sim}{y}-\underset{\sim}{\bar{y}}, z-\bar{z})\big|\!\big|\!\big|_{0,k} \leqslant \frac{C}{\sqrt{m}} \big|\!\big|\!\big|(\underset{\sim}{y}-\underset{\sim}{y}_0, z-z_0)\big|\!\big|\!\big|_{0,k}.$$

Proof. See the proof of Theorem 6.5.8 in [3]. □

Theorem 3 (Convergence of the kth level algorithm). *There exists a constant C, independent of k and m, such that*

$$\big|\!\big|\!\big|(\underset{\sim}{y}, z) - CMG\big(k, (\underset{\sim}{y}_0, z_0), (\underset{\sim}{w}, r)\big)\big|\!\big|\!\big|_{0,k} \leqslant \frac{C}{\sqrt{m}} \big|\!\big|\!\big|(\underset{\sim}{y} - \underset{\sim}{y}_0, z - z_0)\big|\!\big|\!\big|_{0,k}.$$

Proof. See the proof of Theorem 6.5.9 in [3]. □

Remark. Note that the stabilization parameter α has to be bounded by C_I, which is unknown, and to be chosen with the condition $1 - C\sqrt{C_I}\sqrt{\alpha} > 0$, see the proof of Lemma 4, in order to get the convergence of our method.

5. Experimental results

We apply the W-cycle multigrid algorithm to the pure displacement boundary value problem (2) studied in [2]. The domain Ω is the unit square, and the body force $f = (f_1, f_2)$ is taken to be as follows:

$$f_1 = \pi^2 \left[2\sin 2\pi y(-1 + 2\cos 2\pi x) - 0.5\cos\pi(x+y) + \frac{\epsilon}{\epsilon+2}\sin\pi x \sin\pi y\right],$$
$$f_2 = \pi^2 \left[2\sin 2\pi x(1 - 2\cos 2\pi y) - 0.5\cos\pi(x+y) + \frac{\epsilon}{\epsilon+2}\sin\pi x \sin\pi y\right].$$

The exact solution $\underset{\sim}{u} = (u_1, u_2)$ is

$$u_1 = \sin 2\pi y(-1 + \cos 2\pi x) + \frac{\epsilon}{\epsilon+2}\sin\pi x \sin\pi y,$$
$$u_2 = \sin 2\pi x(1 - \cos 2\pi y) + \frac{\epsilon}{\epsilon+2}\sin\pi x \sin\pi y.$$

The programs execute until the discrete L^2 relative error is less than 5% of the initial error. We use the initial iterates $\underset{\sim}{u}^0 = (u_1^0, u_2^0) = (0, 0)$ and $p^0 = 0$. The computations were done in double-precision arithmetic for various α's, smoothing steps and Poisson's ratio ν's. The numbers in the columns represent the number of iterations to achieve an L^2 relative error of less than 5% in the displacement.

Although we have only proven the convergence of the W-cycle multigrid method with $\alpha/4$ at the coarse grid, we give the numerical experiments with the fixed α for all levels in our algorithm and with $\alpha/4$ at the coarse grid. A very attractive feature of using the fixed α for all levels in our algorithm is its inherent simplicity, that is, the bilinear form at the coarse grid is the same form at the fine grid. The numerical experiments show that the number of iterations of W-cycle multigrid method is nearly the same in both cases of the fixed α and changed α. In both cases, we know that the number of iterations for the W-cycle is reduced in half when we take twice as many smoothing steps (m) and cut in half when we have the mesh size by a half. We also observe that our multigrid is robust for the moderate α's in that the convergence is uniform as the parameter, Poisson's ratio ν, goes to $1/2$.

To find an appropriate factor of error reduction after each W-cycle, we have tested several cases with many smoothing steps (m); see Tables 9 and 10. Also, the results, for $\alpha = 0.3, 0.1$, etc., are very similar with Tables 9 and 10. We observe that our method needs many smoothing steps to get a good factor of error reduction after each W-cycle and to reduce the number of iterations.

Table 1
The number of iterations for the W-cycle with $N = 32$ and $\nu = 0.3$

	$\alpha/4$ at the coarse grid				Fixed α for all levels			
	$\alpha = 1$	0.3	0.1	0.05	1	0.3	0.1	0.05
$m = 1$	1096	1084	1081	1080	1180	1097	1088	1085
$m = 2$	548	542	541	540	590	549	544	543
$m = 3$	366	362	361	360	394	366	363	362
$m = 4$	274	271	271	270	295	275	272	272

Table 2
The number of iterations for the W-cycle with $N = 32$ and $\nu = 0.45$

	$\alpha/4$ at the coarse grid				Fixed α for all levels			
	$\alpha = 1$	0.3	0.1	0.05	1	0.3	0.1	0.05
$m = 1$	1084	1079	1082	1083	1177	1101	1092	1086
$m = 2$	542	540	541	542	589	551	546	543
$m = 3$	362	360	361	361	393	367	364	362
$m = 4$	271	270	271	271	295	276	273	272

Table 3
The number of iterations for the W-cycle with $N = 32$ and $\nu = 0.495$

	$\alpha/4$ at the coarse grid				Fixed α for all levels			
	$\alpha = 1$	0.3	0.1	0.05	1	0.3	0.1	0.05
$m = 1$	1091	1102	1113	1116	1174	1100	1094	1094
$m = 2$	546	551	557	558	587	550	547	548
$m = 3$	364	368	371	372	391	367	365	365
$m = 4$	273	276	279	279	294	275	274	274

Table 4
The number of iterations for the W-cycle with $N = 32$ and $\nu = 0.4995$

	$\alpha/4$ at the coarse grid				Fixed α for all levels			
	$\alpha = 1$	0.3	0.1	0.05	1	0.3	0.1	0.05
$m = 1$	1093	1107	1118	1122	1173	1100	1094	1096
$m = 2$	547	554	559	561	587	550	547	561
$m = 3$	365	369	373	375	391	367	365	366
$m = 4$	274	277	280	281	294	275	274	274

Table 5
The number of iterations for the W-cycle with $N = 64$ and $\nu = 0.3$

	$\alpha/4$ at the coarse grid				Fixed α for all levels			
	$\alpha = 1$	0.3	0.1	0.05	1	0.3	0.1	0.05
$m = 1$	554	552	551	551	609	563	556	554
$m = 2$	277	276	276	276	305	282	278	277
$m = 3$	185	184	184	184	203	188	186	185
$m = 4$	139	138	138	138	153	141	139	139

Table 6
The number of iterations for the W-cycle with $N = 64$ and $\nu = 0.45$

	$\alpha/4$ at the coarse grid				Fixed α for all levels			
	$\alpha = 1$	0.3	0.1	0.05	1	0.3	0.1	0.05
$m = 1$	551	552	552	553	611	568	559	555
$m = 2$	276	276	276	277	306	284	280	278
$m = 3$	184	184	184	185	204	190	187	185
$m = 4$	138	138	138	139	153	142	140	139

Table 7
The number of iterations for the W-cycle with $N = 64$ and $\nu = 0.495$

	$\alpha/4$ at the coarse grid				Fixed α for all levels			
	$\alpha = 1$	0.3	0.1	0.05	1	0.3	0.1	0.05
$m = 1$	564	568	570	570	610	569	561	559
$m = 2$	282	284	285	285	305	285	281	280
$m = 3$	188	190	190	190	204	190	187	187
$m = 4$	141	142	143	143	153	143	141	140

Table 8
The number of iterations for the W-cycle with $N = 64$ and $\nu = 0.4995$

	$\alpha/4$ at the coarse grid				Fixed α for all levels			
	$\alpha = 1$	0.3	0.1	0.05	1	0.3	0.1	0.05
$m = 1$	566	571	573	573	610	569	561	560
$m = 2$	283	286	287	287	305	285	281	280
$m = 3$	189	191	191	192	204	190	187	187
$m = 4$	142	143	144	145	153	143	141	140

Table 9
The factor (γ) of error reduction and iteration numbers when $\alpha = 1$, $N = 32$

	$\nu = 0.3$		$\nu = 0.45$		$\nu = 0.495$		$\nu = 0.4995$	
	γ	Iter.	γ	Iter.	γ	Iter.	γ	Iter.
$m = 10$	0.98	110	0.98	109	0.98	110	0.98	110
$m = 50$	0.92	22	0.91	22	0.91	22	0.91	22
$m = 100$	0.85	12	0.83	11	0.83	11	0.83	11

Table 10
The factor (γ) of error reduction and iteration numbers when $\alpha = 1$, $N = 64$

	$\nu = 0.3$		$\nu = 0.45$		$\nu = 0.495$		$\nu = 0.4995$	
	γ	Iter.	γ	Iter.	γ	Iter.	γ	Iter.
$m = 10$	0.97	56	0.97	56	0.97	57	0.97	57
$m = 50$	0.86	12	0.85	12	0.87	12	0.87	12
$m = 100$	0.72	6	0.71	6	0.72	6	0.72	6

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